Alain Guimier

Mathematical derivation of the Lorentz Transformation as an invariant of the d'Alembertian operator.

09/2024

Introduction :

In special relativity, different methods are known for deriving from the hypothesis of the invariance of the numerical speed of light (along with other hypotheses; see for example [1] p.76) : for two observers O *and* O' *in uniform translation at a constant velocity* v *, the nature of the linear operator* M *that relates the coordinates* X *of a point* P *for* O *and those* X' *for* O' *in spacetime is then characterized by* ${}^{t}MGM = G$*, with* $G = (g(i, j))$*,* $1 \leq i, j \leq 4$*, where* $g(1, 1) = 1$ *and* $g(i, i) = -1$ *for* $i \neq 1$*, and* $g(i, j) = 0$ *for* $i \neq j$ *:* M *is named Lorentz transformation .*
Conversely, from this property of M*, one can derive a velocity addition formula (see for example [1] p.68) and show that* c *is invariant :* $\dfrac{(v + c)}{1 + \dfrac{uv}{c^2}} = c$ *, for instance.*
It has long been shown that the d'Alembertian operator is invariant under an operator M *that satisfies* ${}^{t}MGM = G$ *(see for example [1] p.95) .Here, we show the reciprocal : the invariance of the d'Alembertian operator under an operator* M *twice continuously differentiable such as* $M(0) = 0$ *implies that* M *is linear and* ${}^{t}MGM = G$*.*
Since the calculations are very lengthy, they were performed using ***Maple 12*** *software.*
The advantage of deriving from the invariance of the d'Alembertian operator is that it requires less hypotheses, than the derivation based on the invariance of the numerical speed of light.
Another approach to the problem exists in [5].
Many other derivations of the operator M *based on various hypotheses also exist; see [3].*
Notice : we can use (t, x, y, z) *instead of* (x^1, x^2, x^3, x^4) *and vice − versa .*

Reminder :

Definition : *A real square matrix* M *of order* $n > 0$ *is called a Lorentz matrix if it satisfies* ${}^{t}MGM = G$*, with* $G = (g(i, j))$ *where* $g(1, 1) = 1$*,* $g(i, i) = -1$ *for* $i \neq 1$*, and* $g(i, j) = 0$ *for* $i \neq j$*.*
Theorem : *The set* L *of Lorentz matrices* M *of order* $n > 1$ *forms a subgroup of* $GL_n(\mathbb{R})$ *under matrix multiplication due to the following* ***4*** *properties :*
(1) It is indeed a subset of $GL_n(\mathbb{R})$ *because* $\det({}^{t}MGM) = \det(G) = -1 = -(\det(M))^2 \neq 0$*.*
(2) $L \neq \varnothing$ *because both* Id *and* G *are Lorentz matrices.*
(3) Let M *and* N *be two Lorentz matrices : since* ${}^{t}(MN)G(MN) = {}^{t}N({}^{t}MGM)N = G$*,* L *is therefore closed under multiplication.*
(4) Since ${}^{t}MGM = G$*, we have* $G = ({}^{t}M)^{-1}GM^{-1} = {}^{t}(M^{-1})GM^{-1}$*, and* L *is closed under inversion.*
Corollary : *If* M *is a Lorentz matrix, then* ${}^{t}M$ *is also a Lorentz one , specifically :* ${}^{t}MGM = G \Leftrightarrow MG{}^{t}M = G$*.*
Since ${}^{t}MGM = G$*, we have* $({}^{t}MGM)^{-1} = G^{-1} = G = M^{-1}G({}^{t}M)^{-1}$*, thus* $M^{-1} = G{}^{t}MG$*, and* ${}^{t}M = GM^{-1}G$ *is a Lorentz matrix as product of matricesof Lorentz.*
We can notice :

if $\boldsymbol{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$ *is a Lorentz matrix then* $\boldsymbol{M}^{-1} = \begin{bmatrix} m_{1,1} & -m_{2,1} & -m_{3,1} & -m_{4,1} \\ -m_{1,2} & m_{2,2} & m_{3,2} & m_{4,2} \\ -m_{1,3} & m_{2,3} & m_{3,3} & m_{4,3} \\ -m_{1,4} & m_{2,4} & m_{3,4} & m_{4,4} \end{bmatrix}$.

Consequences *of* (4) *on the, coefficients of a matrix of Lorentz* **:**

For example if $n = 2$ *if* $\boldsymbol{M} = \begin{bmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \end{bmatrix}$ $\boldsymbol{MG{}^tM} = \boldsymbol{G}$ *implique que :*

$$\begin{bmatrix} m_{1,1}^2 - m_{1,2}^2 & m_{1,1}m_{2,1} - m_{1,2}m_{2,2} \\ m_{1,1}m_{2,1} - m_{1,2}m_{2,2} & m_{2,1}^2 - m_{2,2}^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and if $\boldsymbol{n} = \boldsymbol{4}$ *:*

$$\begin{gathered}
\big[\big[m_{1,1}^2 - m_{1,2}^2 - m_{1,3}^2 - m_{1,4}^2,\ m_{1,1}m_{2,1} - m_{1,2}m_{2,2} - m_{1,3}m_{2,3} - m_{1,4}m_{2,4},\ m_{1,1}m_{3,1} - m_{1,2}m_{3,2} - m_{1,3}m_{3,3} - m_{1,4}m_{3,4},\ m_{1,1}m_{4,1} \\
- m_{1,2}m_{4,2} - m_{1,3}m_{4,3} - m_{1,4}m_{4,4}\big], \\
\big[m_{1,1}m_{2,1} - m_{1,2}m_{2,2} - m_{1,3}m_{2,3} - m_{1,4}m_{2,4},\ m_{2,1}^2 - m_{2,2}^2 - m_{2,3}^2 - m_{2,4}^2,\ m_{2,1}m_{3,1} - m_{2,2}m_{3,2} - m_{2,3}m_{3,3} - m_{2,4}m_{3,4},\ m_{2,1}m_{4,1} \\
- m_{2,2}m_{4,2} - m_{2,3}m_{4,3} - m_{2,4}m_{4,4}\big], \\
\big[m_{1,1}m_{3,1} - m_{1,2}m_{3,2} - m_{1,3}m_{3,3} - m_{1,4}m_{3,4},\ m_{2,1}m_{3,1} - m_{2,2}m_{3,2} - m_{2,3}m_{3,3} - m_{2,4}m_{3,4},\ m_{3,1}^2 - m_{3,2}^2 - m_{3,3}^2 - m_{3,4}^2,\ m_{3,1}m_{4,1} \\
- m_{3,2}m_{4,2} - m_{3,3}m_{4,3} - m_{3,4}m_{4,4}\big], \\
\big[m_{1,1}m_{4,1} - m_{1,2}m_{4,2} - m_{1,3}m_{4,3} - m_{1,4}m_{4,4},\ m_{2,1}m_{4,1} - m_{2,2}m_{4,2} - m_{2,3}m_{4,3} - m_{2,4}m_{4,4},\ m_{3,1}m_{4,1} - m_{3,2}m_{4,2} - m_{3,3}m_{4,3} \\
- m_{3,4}m_{4,4},\ m_{4,1}^2 - m_{4,2}^2 - m_{4,3}^2 - m_{4,4}^2\big]\big]
\end{gathered}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

and vice versa if the coefficients of the matrix ***M*** *verify these relationships ,*
the matrix ***M*** *is a Lorentz matrix.*

Differential calculus reminder :

Subsequently, to simplify the writings, we will assume that the numerical speed of light $\boldsymbol{c} = \boldsymbol{1}$.

Let $\boldsymbol{f}$ *be a class* $\mathcal{C}^2$ *- function (twice continuously differentiable) with real values :* $\left(x^i\right)_1^4 \to f\left(\left(x^i\right)_1^4\right)$,

$\boldsymbol{M} = \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$ *a constant matrix ,* ***X*** *and* ***X'*** *such that* $\boldsymbol{X' = MX}$.

We notice as $\boldsymbol{X'}=\boldsymbol{MX}=\begin{bmatrix} m_{1,1}x^1+m_{1,2}x^2+m_{1,3}x^3+m_{1,4}x^4 \\ m_{2,1}x^1+m_{2,2}x^2+m_{2,3}x^3+m_{2,4}x^4 \\ m_{3,1}x^1+m_{3,2}x^2+m_{3,3}x^3+m_{3,4}x^4 \\ m_{4,1}x^1+m_{4,2}x^2+m_{4,3}x^3+m_{4,4}x^4 \end{bmatrix}.$

we have $\dfrac{\partial x'^a}{\partial x^b}=m_{a,b}$ *and* $(cf: [1]p.95\ et[4])$ *:*

$$\frac{\partial f(X')}{\partial x^b}=\sum_{a=1}^{4}\frac{\partial f(X')}{\partial x'^a}\ \frac{\partial x'^a}{\partial x^b}=\sum_{a=1}^{4}\frac{\partial f(X')}{\partial x'^a}\ m_{a,b}.\ (\boldsymbol{E})$$

Let's evaluate now $\dfrac{\partial^2 f(X')}{\partial x^{b^2}}$ *:*

$$\frac{\partial 2f(X')}{\partial x^{b2}}=\frac{\partial}{\partial x^b}\left(\frac{\partial f(X)}{\partial x^b}\right)=\frac{\partial}{\partial x^b}\left(\sum_{a=1}^{4}\frac{\partial f(X')}{\partial x'^a}\ m_{a,b}\right)=\sum_{a=1}^{4}\frac{\partial}{\partial x^b}\ \frac{\partial f(X')}{\partial x'^a}\ m_{a,b}.$$

Because of $(\boldsymbol{E})$ **:**

$$\frac{\partial}{\partial x^b}\ \frac{\partial f(X')}{\partial x'^a}=\sum_{c=1}^{4}\frac{\partial}{\partial x'^c}\ \frac{\partial f(X')}{\partial x'^a}m_{c,b}\ \textit{then}\ \frac{\partial 2f(X)}{\partial x^{b2}}=\sum_{a=1}^{4}\sum_{c=1}^{4}\frac{\partial}{\partial x'^c}\ \frac{\partial f(X')}{\partial x'^a}m_{c,b}\cdot m_{a,b}\ .$$

$$\frac{\partial 2f(X')}{\partial x^{12}}-\frac{\partial 2f(X')}{\partial x^{22}}-\frac{\partial 2f(X')}{\partial x^{32}}-\frac{\partial 2f(X')}{\partial x^{42}}=\sum_{a=1}^{4}\sum_{c=1}^{4}\frac{\partial}{\partial x'^c}\ \frac{\partial f(X')}{\partial x'^a}m_{c,1}\cdot m_{a,1}-\sum_{i=2}^{4}\sum_{a=1}^{4}\sum_{c=1}^{4}\frac{\partial}{\partial x'^c}\ \frac{\partial f(X')}{\partial x'^a}m_{c,i}\cdot m_{a,i}.$$

Theorem *: Let be* $\boldsymbol{M}=\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}$ *a matrix* **,**

$\boldsymbol{X'}=\boldsymbol{MX}$*, with* $\boldsymbol{X'}=\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix}$ *and* $\boldsymbol{X}=\begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}.$

Let $\boldsymbol{f}$ *be a class* C^2 *function with real values :* $\boldsymbol{f}:(t,x,y,z)\to \boldsymbol{f}(t,x,y,z)$

and $\Box$ *the operator :*

$$\Box:(f,X))\to\frac{\partial^2 f}{\partial t^2}-\frac{\partial^2 f}{\partial x^2}-\frac{\partial^2 f}{\partial y^2}-\frac{\partial^2 f}{\partial z^2}\qquad :$$

i.e. $\Box(f)(t,x,y,z)=\dfrac{\partial^2 f(t,x,y,z)}{\partial t^2}-\dfrac{\partial^2 f(t,x,y,z)}{\partial x^2}-\dfrac{\partial^2 f(t,x,y,z)}{\partial y^2}-\dfrac{\partial^2 f(t,x,y,z)}{\partial z^2}\ .$

If $\forall\,(\boldsymbol{t, x, y, z})\ \forall \boldsymbol{f} \in \mathcal{C}^2 : \square\,(\boldsymbol{f} \circ \boldsymbol{M})\,(\boldsymbol{t, x, y, z}) = \square\,(\boldsymbol{f})\ (\boldsymbol{t', x', y', z'})$ *with* $\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = M \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$

then ***M*** *is a Lorentz matrix and conversely.*
This condition can be rewritten as :

$$\frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial t^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial x^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial y^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial z^2}$$
$$= \frac{\partial^2 f(t', x', y', z')}{\partial t'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial x'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial y'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial z'^2}$$

Proof :
[***I***] *First method :*
We use ***Maple 12*** *'commands to evaluate :*

$$\frac{\partial^2}{\partial t^2} f(t', x', y', z') - \frac{\partial^2}{\partial x^2} f(t', x', y', z') - \frac{\partial^2}{\partial y^2} f(t', x', y', z') - \frac{\partial^2}{\partial z^2} f(t', x', y', z') :$$

$$restart : g := (t, x, y, z) \to \left(\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} . \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \right) : h := M \to (M[1], M[2], M[3], M[4]) :$$

$$A := simplify\left(\left(\frac{\partial^2}{\partial t\, \partial t} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial x\, \partial x} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial y\, \partial y} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial z\, \partial z} f(h(g(t, x, y, z))) \right) \right) \quad :$$

$$a := algsubs(m_{1,1} \cdot t + m_{1,2} \cdot x + m_{1,3} \cdot y + m_{1,4} \cdot z = t_p, A) :$$
$$b := algsubs(m_{2,1} \cdot t + m_{2,2} \cdot x + m_{2,3} \cdot y + m_{2,4} \cdot z = x_p, a) :$$
$$c := algsubs(m_{3,1} \cdot t + m_{3,2} \cdot x + m_{3,3} \cdot y + m_{3,4} \cdot z = y_p, b) :$$
$$e1 := algsubs(m_{4,1} \cdot t + m_{4,2} \cdot x + m_{4,3} \cdot y + m_{4,4} \cdot z = z_p, c) :$$

R := e1 :
for i from 1 to 4 do
for j from 1 to 4 do
$$R := collect(R, D_{i,j}(f)\,(t_p, x_p, y_p, z_p)) :$$
end do
end do :
R;

$$\left(-m_{4,3}^2 - m_{4,4}^2 - m_{4,2}^2 + m_{4,1}^2 \right) D_{4,4}(f)\,(t_p, x_p, y_p, z_p) + \left(-2\, m_{3,4} m_{4,4} - 2\, m_{3,2} m_{4,2} - 2\, m_{3,3} m_{4,3} + 2\, m_{3,1} m_{4,1} \right) D_{3,4}(f)\,(t_p, x_p, y_p, z_p) + \left(m_{3,1}^2 - m_{3,4}^2 - m_{3,3}^2 - m_{3,2}^2 \right) D_{3,3}(f)\,(t_p, x_p, y_p, z_p) + \left(2\, m_{4,1} m_{2,1} - 2\, m_{4,2} m_{2,2} - 2\, m_{4,3} m_{2,3} - 2\, m_{4,4} m_{2,4} \right) D_{2,4}(f)\,(t_p, x_p, y_p, z_p) + \left(2\, m_{3,1} m_{2,1} - 2\, m_{3,2} m_{2,2} - 2\, m_{3,4} m_{2,4} - 2\, m_{3,3} m_{2,3} \right) D_{2,3}(f)\,(t_p, x_p, y_p, z_p) + \left(-m_{2,3}^2 + m_{2,1}^2 - m_{2,4}^2 - m_{2,2}^2 \right) D_{2,2}(f)\,(t_p, x_p, y_p, z_p) + \left(2\, m_{4,1} m_{1,1} - 2\, m_{4,4} m_{1,4} - 2\, m_{4,3} m_{1,3} - 2\, m_{4,2} m_{1,2} \right) D_{1,4}(f)\,(t_p, x_p, y_p, z_p) + \left(-m_{1,2}^2 - m_{1,3}^2 - m_{1,4}^2 + m_{1,1}^2 \right) D_{1,1}(f)\,(t_p, x_p, y_p, z_p)$$

$$+ (2\, m_{2,1} m_{1,1} - 2\, m_{2,4} m_{1,4} - 2\, m_{2,3} m_{1,3} - 2\, m_{2,2} m_{1,2})\, D_{1,2}(f)\,(t_p, x_p, y_p, z_p) + ($$
$$-2\, m_{3,3} m_{1,3} - 2\, m_{3,4} m_{1,4} - 2\, m_{3,2} m_{1,2} + 2\, m_{3,1} m_{1,1})\, D_{1,3}(f)\,(t_p, x_p, y_p, z_p)$$

We use t_p, x_p, y_p, z_p *instead of* t', x', y', z' *because of* **Maple12 :** " ' "*is considered as a derivation .*

As $D_{i,j} = D_{j,i}$ *their multiplying factors are gathered together .*

then ($\boldsymbol{E}$) $$R = \frac{\partial 2 f(X')}{\partial x^{12}} - \frac{\partial 2 f(X')}{\partial x^{22}} - \frac{\partial 2 f(X')}{\partial x^{32}} - \frac{\partial 2 f(X')}{\partial x^{42}} =$$

$$(m_{4,1}^2 - m_{4,2}^2 - m_{4,3}^2 - m_{4,4}^2)\, D_{4,4}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{3,3}\, m_{4,3} + 2\, m_{3,1}\, m_{4,1} - 2\, m_{3,4}\, m_{4,4} - 2\, m_{3,2}\, m_{4,2})\, D_{3,4}(f)\,(t_p, x_p, y_p, z_p) \qquad \textbf{(1)}$$
$$+ (-m_{3,3}^2 + m_{3,1}^2 - m_{3,4}^2 - m_{3,2}^2)\, D_{3,3}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{4,2}\, m_{2,2} - 2\, m_{4,3}\, m_{2,3} + 2\, m_{4,1}\, m_{2,1} - 2\, m_{4,4}\, m_{2,4})\, D_{2,4}(f)\,(t_p,$$
$$x_p, y_p, z_p) + (-2\, m_{3,2}\, m_{2,2} - 2\, m_{3,3}\, m_{2,3} + 2\, m_{3,1}\, m_{2,1} - 2\, m_{3,4}\, m_{2,4})\, D_{2,3}(f)\,(t_p, x_p, y_p, z_p) + (-m_{2,3}^2 + m_{2,1}^2 - m_{2,2}^2 -$$
$$m_{2,4}^2)\, D_{2,2}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{4,2}\, m_{1,2} - 2\, m_{4,3}\, m_{1,3} + 2\, m_{4,1}\, m_{1,1} - 2\, m_{4,4}\, m_{1,4})\, D_{1,4}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{2,2}\, m_{1,2}$$
$$- 2\, m_{2,3}\, m_{1,3} + 2\, m_{2,1}\, m_{1,1} - 2\, m_{2,4}\, m_{1,4})\, D_{1,2}(f)\,(t_p, x_p, y_p, z_p) + (-m_{1,4}^2 + m_{1,1}^2 - m_{1,3}^2 - m_{1,2}^2)\, D_{1,1}(f)\,(t_p, x_p, y_p, z_p) + ($$
$$-2\, m_{3,2}\, m_{1,2} - 2\, m_{3,3}\, m_{1,3} - 2\, m_{3,4}\, m_{1,4} + 2\, m_{3,1}\, m_{1,1})\, D_{1,3}(f)\,(t_p, x_p, y_p, z_p)$$

As $$\frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial t^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial x^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial y^2} - \frac{\partial^2 (f \circ M)(t, x, y, z)}{\partial z^2}$$
$$= \frac{\partial^2 f(t', x', y', z')}{\partial t'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial x'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial y'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial z'^2}.$$

then $$R - \frac{\partial^2 f(t', x', y', z')}{\partial t'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial x'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial y'^2} - \frac{\partial^2 f(t', x', y', z')}{\partial z'^2} = 0 \ .$$

$$\sum = (m_{4,1}^2 - m_{4,2}^2 - m_{4,3}^2 - m_{4,4}^2 + 1)\, D_{4,4}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{3,3}\, m_{4,3} + 2\, m_{3,1}\, m_{4,1} - 2\, m_{3,4}\, m_{4,4} - 2\, m_{3,2}\, m_{4,2})\, D_{3,4}(f)\,(t_p, x_p, y_p, z_p)$$
$$+ (-m_{3,3}^2 + m_{3,1}^2 - m_{3,4}^2 - m_{3,2}^2 + 1)\, D_{3,3}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{4,2}\, m_{2,2} - 2\, m_{4,3}\, m_{2,3} + 2\, m_{4,1}\, m_{2,1} - 2\, m_{4,4}\, m_{2,4})\, D_{2,4}(f)\,(t_p, x_p, y_p,$$
$$z_p) + (-2\, m_{3,2}\, m_{2,2} - 2\, m_{3,3}\, m_{2,3} + 2\, m_{3,1}\, m_{2,1} - 2\, m_{3,4}\, m_{2,4})\, D_{2,3}(f)\,(t_p, x_p, y_p, z_p) + (-m_{2,3}^2 + m_{2,1}^2 - m_{2,2}^2 - m_{2,4}^2 + 1)\, D_{2,2}(f)\,(t_p, x_p,$$
$$y_p, z_p) + (-2\, m_{4,2}\, m_{1,2} - 2\, m_{4,3}\, m_{1,3} + 2\, m_{4,1}\, m_{1,1} - 2\, m_{4,4}\, m_{1,4})\, D_{1,4}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{2,2}\, m_{1,2} - 2\, m_{2,3}\, m_{1,3} + 2\, m_{2,1}\, m_{1,1}$$
$$- 2\, m_{2,4}\, m_{1,4})\, D_{1,2}(f)\,(t_p, x_p, y_p, z_p) + (-m_{1,4}^2 + m_{1,1}^2 - m_{1,3}^2 - m_{1,2}^2 - 1)\, D_{1,1}(f)\,(t_p, x_p, y_p, z_p) + (-2\, m_{3,2}\, m_{1,2} - 2\, m_{3,3}\, m_{1,3}$$
$$- 2\, m_{3,4}\, m_{1,4} + 2\, m_{3,1}\, m_{1,1})\, D_{1,3}(f)\,(t_p, x_p, y_p, z_p) = 0 \,.$$

Let's consider now only the functions $f_u : (t, x, y, z) \rightarrow \frac{1}{2} \cdot t^2 \cdot u_1 + t \cdot x \cdot u_2 + t \cdot y \cdot u_3 + t \cdot z \cdot u_4 + \frac{1}{2} \cdot x^2 \cdot u_5 + x \cdot y \cdot u_6 + x \cdot z \cdot u_7 + \frac{1}{2} \cdot y^2 \cdot u_8 + y \cdot z \cdot u_9 + \frac{1}{2} \cdot z^2 \cdot u_{10}$ *where* $u = (u_1, \ldots, u_{10})$ $u_i \in \mathbb{R}$.

For these functions we have **:**

$D_{1,1}(f_u) = u_1$, $D_{1,2}(f_u) = u_2$, ..., $D_{4,4}(f_u) = u_{10}$.

$\sum$ *is a sum of* **10** *terms, each being the product of a factor* $D_{i,j}(f)\,(t_p, x_p, y_p, z_p)$ *and a polynomial in* $m_{i,j}$.

We can rewrite Σ *as the scalar product of two vectors of size* **10** *:*

$Z1 = {}^{t}\left[D_{11}(f)\left(t_p, x_p, y_p, z_p\right), D_{12}(f)\left(t_p, x_p, y_p, z_p\right)........., D_{44}(f)\left(t_p, x_p, y_p, z_p\right)\right]$

If $\mathbf{D}_{ij}$ *is already in the list, we omit* $\mathbf{D}_{j,i}$ *and*

$\mathbf{Z2}$ *is composed of the terms in* $\mathbf{m}_{i,j}$, *written such that* $\Sigma = {}^{t}Z2.Z1$

. *The functions* $\mathbf{f}_u$ *defined above satisfy :*

$$\frac{\partial^2}{\partial t^2} f_u\left(t_p, x_p, y_p, z_p\right) = u_1, \frac{\partial}{\partial t}\frac{\partial}{\partial x} f_u\left(t_p, x_p, y_p, z_p\right) = u_2, \frac{\partial}{\partial t}\frac{\partial}{\partial y} f_u\left(t_p, x_p, y_p, z_p\right) = u_3,$$

$$\frac{\partial}{\partial t}\frac{\partial}{\partial z} f_u\left(t_p, x_p, y_p, z_p\right) = u_4, \frac{\partial^2}{\partial x^2} f_u\left(t_p, x_p, y_p, z_p\right) = u_5, \frac{\partial}{\partial x}\frac{\partial}{\partial y} f_u\left(t_p, x_p, y_p, z_p\right) = u_6,....$$

Given **10** *reals* $\left(u_i\right)_{1 \le i \le 10}$ *from the above there exists at least one function* $\mathbf{f}_u$ *of class* C^2 *such that*

$Z1 = {}^{t}\left[u_1, ..., u_{10}\right]$. *So* $\sum = {}^{t}Z2.Z1 = 0$ *for all* $\mathbf{u}$ *then* $\mathbf{Z2 = 0}$.

Returning to ***M***, *remembering that :*

$$\left[\left[m_{1,1}^2 - m_{1,2}^2 - m_{1,3}^2 - m_{1,4}^2, m_{1,1} m_{2,1} - m_{1,2} m_{2,2} - m_{1,3} m_{2,3} - m_{1,4} m_{2,4}, m_{1,1} m_{3,1} - m_{1,2} m_{3,2} - m_{1,3} m_{3,3} - m_{1,4} m_{3,4}, m_{1,1} m_{4,1} - m_{1,2} m_{4,2} - m_{1,3} m_{4,3} - m_{1,4} m_{4,4}\right],\right.$$
$$\left[m_{1,1} m_{2,1} - m_{1,2} m_{2,2} - m_{1,3} m_{2,3} - m_{1,4} m_{2,4}, m_{2,1}^2 - m_{2,2}^2 - m_{2,3}^2 - m_{2,4}^2, m_{2,1} m_{3,1} - m_{2,2} m_{3,2} - m_{2,3} m_{3,3} - m_{2,4} m_{3,4}, m_{2,1} m_{4,1} - m_{2,2} m_{4,2} - m_{2,3} m_{4,3} - m_{2,4} m_{4,4}\right],$$
$$\left[m_{1,1} m_{3,1} - m_{1,2} m_{3,2} - m_{1,3} m_{3,3} - m_{1,4} m_{3,4}, m_{2,1} m_{3,1} - m_{2,2} m_{3,2} - m_{2,3} m_{3,3} - m_{2,4} m_{3,4}, m_{3,1}^2 - m_{3,2}^2 - m_{3,3}^2 - m_{3,4}^2, m_{3,1} m_{4,1} - m_{3,2} m_{4,2} - m_{3,3} m_{4,3} - m_{3,4} m_{4,4}\right],$$
$$\left.\left[m_{1,1} m_{4,1} - m_{1,2} m_{4,2} - m_{1,3} m_{4,3} - m_{1,4} m_{4,4}, m_{2,1} m_{4,1} - m_{2,2} m_{4,2} - m_{2,3} m_{4,3} - m_{2,4} m_{4,4}, m_{3,1} m_{4,1} - m_{3,2} m_{4,2} - m_{3,3} m_{4,3} - m_{3,4} m_{4,4}, m_{4,1}^2 - m_{4,2}^2 - m_{4,3}^2 - m_{4,4}^2\right]\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

characterizes a Lorentz matrix, as $\mathbf{Z2 = 0}$ *we conclude that* ***M*** *is a Lorentz matrix.*

[***II***] *Second method :*

We start from the formula proved in the reminder of differential calculation :

$$\frac{\partial^2 f(X')}{\partial x^{12}} - \frac{\partial^2 f(X')}{\partial x^{22}} - \frac{\partial^2 f(X')}{\partial x^{32}} - \frac{\partial^2 f(X')}{\partial x^{42}} = \sum_{a=1}^{4}\sum_{c=1}^{4} \frac{\partial}{\partial x'^{c}} \frac{\partial f(X')}{\partial x^{a}} m_{c,1} \cdot m_{a,1} - \sum_{i=2}^{4}\sum_{a=1}^{4}\sum_{c=1}^{4} \frac{\partial}{\partial x'^{c}} \frac{\partial f(X')}{\partial x^{a}} m_{c,i} \cdot m_{a,i}.$$

To do the calculation with ***Maple 12*** *we have the commands :*

$$M := \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix}; F := \begin{bmatrix} D_{1,1}(f)\left(t_p, x_p, y_p, z_p\right) & D_{1,2}(f)\left(t_p, x_p, y_p, z_p\right) & D_{1,3}(f)\left(t_p, x_p, y_p, z_p\right) & D_{1,4}(f)\left(t_p, x_p, y_p, z_p\right) \\ D_{1,2}(f)\left(t_p, x_p, y_p, z_p\right) & D_{2,2}(f)\left(t_p, x_p, y_p, z_p\right) & D_{2,3}(f)\left(t_p, x_p, y_p, z_p\right) & D_{2,4}(f)\left(t_p, x_p, y_p, z_p\right) \\ D_{1,3}(f)\left(t_p, x_p, y_p, z_p\right) & D_{2,3}(f)\left(t_p, x_p, y_p, z_p\right) & D_{3,3}(f)\left(t_p, x_p, y_p, z_p\right) & D_{3,4}(f)\left(t_p, x_p, y_p, z_p\right) \\ D_{1,4}(f)\left(t_p, x_p, y_p, z_p\right) & D_{2,4}(f)\left(t_p, x_p, y_p, z_p\right) & D_{3,4}(f)\left(t_p, x_p, y_p, z_p\right) & D_{4,4}(f)\left(t_p, x_p, y_p, z_p\right) \end{bmatrix}:$$

```
for k from 1  to 4 do
  X1 := 0 :
 for i from 1  to 4 do
for j  from 1  to 4 do
X1 := X1  + F[j, i]·M[j, k]·M[i, k] :
  end do;
 end do;
 X[k] := X1 :
end do:
 X1 := X[1] :
 for k from 2  to 4 do
 X1 := X1 -X[k];
 end do:
 X1;
for i from 1  to 4 do
for j from 1  to 4 do
 X1 := collect(X1, F[j, i]) :
 end do : end do :
 X1;
```

We obtain :

$$\begin{aligned}&\left(m_{4,1}^2-m_{4,3}^2-m_{4,2}^2-m_{4,4}^2\right) D_{4,4}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-2 m_{4,3} m_{3,3}-2 m_{4,2} m_{3,2}+2 m_{4,1} m_{3,1}\right.\\&\quad\left.-2 m_{4,4} m_{3,4}\right) D_{3,4}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-m_{3,4}^2-m_{3,2}^2+m_{3,1}^2-m_{3,3}^2\right) D_{3,3}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(-2 m_{4,2} m_{2,2}+2 m_{4,1} m_{2,1}-2 m_{4,3} m_{2,3}-2 m_{4,4} m_{2,4}\right) D_{2,4}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(2 m_{3,1} m_{2,1}-2 m_{3,3} m_{2,3}-2 m_{3,4} m_{2,4}-2 m_{3,2} m_{2,2}\right) D_{2,3}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(2 m_{2,1} m_{1,1}-2 m_{2,4} m_{1,4}-2 m_{2,3} m_{1,3}-2 m_{2,2} m_{1,2}\right) D_{1,2}(f)\left(t_p, x_p, y_p, z_p\right)+\left(\right.\\&\quad\left.-2 m_{3,4} m_{1,4}+2 m_{3,1} m_{1,1}-2 m_{3,3} m_{1,3}-2 m_{3,2} m_{1,2}\right) D_{1,3}(f)\left(t_p, x_p, y_p, z_p\right)+\left(m_{2,1}^2-m_{2,3}^2\right.\\&\quad\left.-m_{2,2}^2-m_{2,4}^2\right) D_{2,2}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-m_{1,4}^2-m_{1,3}^2-m_{1,2}^2+m_{1,1}^2\right) D_{1,1}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(-2 m_{4,3} m_{1,3}-2 m_{4,2} m_{1,2}-2 m_{4,4} m_{1,4}+2 m_{4,1} m_{1,1}\right) D_{1,4}(f)\left(t_p, x_p, y_p, z_p\right).\end{aligned}$$

We find the same calculated result in the first part and we finish in the same way.

The reciprocal is obvious :
As previously ,

$$\begin{aligned}&\frac{\partial^2}{\partial t^2} f(t', x', y', z')-\frac{\partial^2}{\partial x^2} f(t', x', y', z')-\frac{\partial^2}{\partial y^2} f(t', x', y', z')-\frac{\partial^2}{\partial z^2} f(t', x', y', z')\\&=\left(m_{4,1}^2-m_{4,3}^2-m_{4,2}^2-m_{4,4}^2\right) D_{4,4}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-2 m_{4,3} m_{3,3}-2 m_{4,2} m_{3,2}+2 m_{4,1} m_{3,1}\right.\\&\quad\left.-2 m_{4,4} m_{3,4}\right) D_{3,4}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-m_{3,4}^2-m_{3,2}^2+m_{3,1}^2-m_{3,3}^2\right) D_{3,3}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(-2 m_{4,2} m_{2,2}+2 m_{4,1} m_{2,1}-2 m_{4,3} m_{2,3}-2 m_{4,4} m_{2,4}\right) D_{2,4}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(2 m_{3,1} m_{2,1}-2 m_{3,3} m_{2,3}-2 m_{3,4} m_{2,4}-2 m_{3,2} m_{2,2}\right) D_{2,3}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(2 m_{2,1} m_{1,1}-2 m_{2,4} m_{1,4}-2 m_{2,3} m_{1,3}-2 m_{2,2} m_{1,2}\right) D_{1,2}(f)\left(t_p, x_p, y_p, z_p\right)+\left(\right.\\&\quad\left.-2 m_{3,4} m_{1,4}+2 m_{3,1} m_{1,1}-2 m_{3,3} m_{1,3}-2 m_{3,2} m_{1,2}\right) D_{1,3}(f)\left(t_p, x_p, y_p, z_p\right)+\left(m_{2,1}^2-m_{2,3}^2\right.\\&\quad\left.-m_{2,2}^2-m_{2,4}^2\right) D_{2,2}(f)\left(t_p, x_p, y_p, z_p\right)+\left(-m_{1,4}^2-m_{1,3}^2-m_{1,2}^2+m_{1,1}^2\right) D_{1,1}(f)\left(t_p, x_p, y_p, z_p\right)\\&\quad+\left(-2 m_{4,3} m_{1,3}-2 m_{4,2} m_{1,2}-2 m_{4,4} m_{1,4}+2 m_{4,1} m_{1,1}\right) D_{1,4}(f)\left(t_p, x_p, y_p, z_p\right)\\&=\frac{\partial^2 f(t', x', y', z')}{\partial t'^2}-\frac{\partial^2 f(t', x', y', z')}{\partial x'^2}-\frac{\partial^2 f(t', x', y', z')}{\partial y'^2}-\frac{\partial^2 f(t', x', y', z')}{\partial z'^2}\end{aligned}$$

*if **M** is a matrix of Lorentz .*

Numerical examples :

$$M := \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} -2 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} : G := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} : M^{\%T}.G.M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$M.G.M^{\%T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, IM := G.M^{\%T}.G, IM = \begin{bmatrix} -\sqrt{2} & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ 0 & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix},$$

$$IM.M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, g := (t, x, y, z) \to \left(M. \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \right) : h := M \to (M[1], M[2], M[3], M[4]) :$$

$h(g(t, x, y, z));$ $-\sqrt{2}\, t + \frac{1}{2}\sqrt{2}\, x - \frac{1}{2}\sqrt{2}\, z, -\frac{1}{2}\sqrt{2}\, t + \frac{1}{2}\sqrt{2}\, x + \frac{1}{2}\sqrt{2}\, y - \frac{1}{2}\sqrt{2}\, z, \frac{1}{2}\sqrt{2}\, t - \frac{1}{2}\sqrt{2}\, x + \frac{1}{2}\sqrt{2}\, y + \frac{1}{2}\sqrt{2}\, z,$

$\frac{1}{2}\sqrt{2}\, x + \frac{1}{2}\sqrt{2}\, z$

$$f := (t, x, y, z) \to y - x^3 + 3 \cdot t^2 + x \cdot t + x \cdot y + z^4 + \sin(x) :$$

$$A := \mathrm{simplify}\left(\left(\frac{\partial^2}{\partial t\, \partial t} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial x\, \partial x} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial y\, \partial y} f(h(g(t, x, y, z))) \right) - \left(\frac{\partial^2}{\partial z\, \partial z} f(h(g(t, x, y, z))) \right) \right);$$

$$6 - 6\, x^2 - 12\, x\, z - 6\, z^2 - 3\sqrt{2}\, t + 3\sqrt{2}\, x - 3\sqrt{2}\, z + 3\sqrt{2}\, y - \sin\left(\frac{1}{2}\sqrt{2}\, t - \frac{1}{2}\sqrt{2}\, x - \frac{1}{2}\sqrt{2}\, y + \frac{1}{2}\sqrt{2}\, z \right)$$

$$B := \mathrm{simplify}\left(\left(\frac{\partial^2}{\partial t\, \partial t} f(t, x, y, z) \right) - \left(\frac{\partial^2}{\partial x\, \partial x} f(t, x, y, z) \right) - \left(\frac{\partial^2}{\partial y\, \partial y} f(t, x, y, z) \right) - \left(\frac{\partial^2}{\partial z\, \partial z} f(t, x, y, z) \right) \right);$$

$$6 + 6\, x + \sin(x) - 12\, z^2$$

If you replace t by t', ...

$$6 + 6 \cdot \left(-\frac{1}{2}\sqrt{2}\, t + \frac{1}{2}\sqrt{2}\, x + \frac{1}{2}\sqrt{2}\, y - \frac{1}{2}\sqrt{2}\, z \right) + \sin\left(-\frac{1}{2}\sqrt{2}\, t + \frac{1}{2}\sqrt{2}\, x + \frac{1}{2}\sqrt{2}\, y - \frac{1}{2}\sqrt{2}\, z \right)$$
$$- 12 \cdot \left(\frac{1}{2} \cdot \sqrt{2} \cdot x + \frac{1}{2} \cdot \sqrt{2} \cdot z \right)^2 :$$

$\mathrm{simplify}(\%);$

$$6 - 6\, x^2 - 12\, x\, z - 6\, z^2 - 3\sqrt{2}\, t + 3\sqrt{2}\, x - 3\sqrt{2}\, z + 3\sqrt{2}\, y - \sin\left(\frac{1}{2}\sqrt{2}\, t - \frac{1}{2}\sqrt{2}\, x - \frac{1}{2}\sqrt{2}\, y + \frac{1}{2}\sqrt{2}\, z \right)$$

Case : M is not linear :

Now we assume only that $\boldsymbol{M}$ *is an operator twice continuously differentiable such that* $\boldsymbol{M}(\boldsymbol{0},\boldsymbol{0})=(\boldsymbol{0},\boldsymbol{0})$ *we can prove that is matrix of Lorentz .*

To do this and to limit the length of the formulas we limit our proof to $\boldsymbol{n=2}$ *.*

Hypothesis :

$$\forall\,(t,x)\ \forall f\in\mathcal{C}^2:\square(f\circ M)\,(t,x)=\square(f)\,(t',x')\ \text{ with }\begin{bmatrix}t'\\x'\end{bmatrix}=M\begin{bmatrix}t\\x\end{bmatrix}.$$

This condition can be rewritten as :

$$\frac{\partial^2(f\circ M)(t,x)}{\partial t^2}-\frac{\partial^2(f\circ M)(t,x)}{\partial x^2}=\frac{\partial^2 f(t',x')}{\partial t'^2}-\frac{\partial^2 f(t',x')}{\partial x'^2}\quad\text{with }(t',x')=M(t,x)\ .$$

Proof :

As previously let's use ***Maple 12*** *with the commands :*

restart :

$h := M\to(M[1],M[2])$ **:**

$A := simplify\left(\left(\frac{\partial 2}{\partial t\ \partial t}f(h(M(t,x)))\right)-\left(\frac{\partial 2}{\partial x\ \partial x}f(h(M(t,x)))\right)\right)\quad;$

$R := A$ **:**

for i from 1 to 2 do

for j from 1 to 2 do

$R := collect\left(R, D_{i,j}(f)\,\left(M(t,x)_1, M(t,x)_2\right)\right)$ **:**

end do

end do :

R;

We obtain :

$$\frac{\partial^2 f(t',x')}{\partial t'^2}-\frac{\partial^2 f(t',x')}{\partial x'^2}=\frac{\partial^2}{\partial t^2}f\left(M(t,x)_1,M(t,x)_2\right)-\frac{\partial^2}{\partial x^2}f\left(M(t,x)_1,M(t,x)_2\right)=$$

$$\left(\left(\frac{\partial}{\partial t}M(t,x)_2\right)^2-\left(\frac{\partial}{\partial x}M(t,x)_2\right)^2\right)D_{2,2}(f)\left(M(t,x)_1,M(t,x)_2\right)+\left(2\left(\frac{\partial}{\partial t}M(t,x)_1\right)\left(\frac{\partial}{\partial t}M(t,x)_2\right)-2\left(\frac{\partial}{\partial x}M(t,x)_1\right)\left(\frac{\partial}{\partial x}M(t,x)_2\right)\right)D_{1,2}(f)\left(M(t,x)_1,M(t,x)_2\right)$$
$$+\left(\left(\frac{\partial}{\partial t}M(t,x)_1\right)^2-\left(\frac{\partial}{\partial x}M(t,x)_1\right)^2\right)D_{1,1}(f)\left(M(t,x)_1,M(t,x)_2\right)+D_2(f)\left(M(t,x)_1,M(t,x)_2\right)\left(\left(\frac{\partial^2}{\partial t^2}M(t,x)_2\right)-\left(\frac{\partial^2}{\partial x^2}M(t,x)_2\right)\right)+D_1(f)\left(M(t,x)_1,M(t,x)_2\right)\left(\left(\frac{\partial^2}{\partial t^2}M(t,x)_1\right)-\left(\frac{\partial^2}{\partial x^2}M(t,x)_1\right)\right)$$

Notice : $D_{1,2}(f)=D_{2,1}(f)$ *.*

Let's consider the functions $f_{\alpha,\beta}$ *of the form* $f_{\alpha,\beta}(t,x)=\alpha\cdot t+\beta\cdot x$ *we have* **:**

$$\frac{\partial^2}{\partial t^2} f_{\alpha,\beta}(t',x') - \frac{\partial^2}{\partial x^2} f_{\alpha,\beta}(t',x') = 0,\ D_{i,j}\left(f_{\alpha,\beta}\right) = 0,\ D_1\left(f_{\alpha,\beta}\right) = \alpha \text{ and } D_2\left(f_{\alpha,\beta}\right) = \beta$$

. Now we have :

$$0 = \beta\left(\frac{\partial^2}{\partial t^2} M(t,x)_2 - \frac{\partial^2}{\partial x^2} M(t,x)_2\right) + \alpha\left(\frac{\partial^2}{\partial t^2} M(t,x)_1 - \frac{\partial^2}{\partial x^2} M(t,x)_1\right)\ \forall\ \alpha \text{ and } \beta.$$

Thus :

$$\frac{\partial^2}{\partial t^2} M(t,x)_1 - \frac{\partial^2}{\partial x^2} M(t,x)_1 = 0 \text{ et } \frac{\partial^2}{\partial t^2} M(t,x)_2 - \frac{\partial^2}{\partial x^2} M(t,x)_2 = 0$$

Then (Wave equation) :

$$M(t,x)_1 = F1(t+x) + F2(t-x) \text{ et } M(t,x)_2 = G1(t+x) + G2(t-x) \qquad (E)$$

with arbitry functions **F1, F2 , G1, G2** *of class* $\mathcal{C}^2$.

If **M** *is a matrix ,for example :*

$$M(t,x)_1 = m_{1,1}\,t + m_{1,2}\,x = \frac{\left(m_{1,1} + m_{1,2}\right)}{2}\cdot(t+x) + \frac{\left(m_{1,1} - m_{1,2}\right)}{2}\cdot(t-x).$$

$$M(t,x)_2 = m_{2,1}\,t + m_{2,2}\,x = \frac{\left(m_{2,1} + m_{2,2}\right)}{2}\cdot(t+x) + \frac{\left(m_{2,1} - m_{2,2}\right)}{2}\cdot(t-x)$$

Now if we restrict to the $M(t,x)$ *of the form above* (E) *we have :*

$$\frac{\partial^2}{\partial t^2} f\left(M(t,x)_1, M(t,x)_2\right) - \frac{\partial^2}{\partial x^2} f\left(M(t,x)_1, M(t,x)_2\right) =$$
$$\left(\left(\frac{\partial}{\partial t} M(t,x)_2\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_2\right)^2\right) D_{2,2}(f)\left(M(t,x)_1, M(t,x)_2\right) + \left(2\left(\frac{\partial}{\partial t} M(t,x)_1\right)\left(\frac{\partial}{\partial t} M(t,x)_2\right) - 2\left(\frac{\partial}{\partial x} M(t,x)_1\right)\left(\frac{\partial}{\partial x} M(t,x)_2\right)\right) D_{1,2}(f)\left(M(t,x)_1, M(t,x)_2\right)$$
$$+ \left(\left(\frac{\partial}{\partial t} M(t,x)_1\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_1\right)^2\right) D_{1,1}(f)\left(M(t,x)_1, M(t,x)_2\right).$$

Let's consider the functions $f_{\alpha,\beta,\gamma}(t,x) = \frac{1}{2}\cdot\alpha\cdot t^2 + \beta\cdot t\cdot x + \frac{1}{2}\cdot\gamma\cdot x^2$ *:*

We have :

$$\frac{\partial^2}{\partial t'^2} f_{\alpha,\beta,\gamma}(t',x') - \frac{\partial^2}{\partial x'^2} f_{\alpha,\beta,\gamma}(t',x') = \alpha - \gamma$$
$$= \frac{\partial^2}{\partial t^2} f_{\alpha,\beta,\gamma}\left(M(t,x)_1, M(t,x)_2\right) - \frac{\partial^2}{\partial x^2} f_{\alpha,\beta,\gamma}\left(M(t,x)_1, M(t,x)_2\right)$$
$$= \left(\left(\frac{\partial}{\partial t} M(t,x)_2\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_2\right)^2\right)\gamma + \left(2\left(\frac{\partial}{\partial t} M(t,x)_1\right)\left(\frac{\partial}{\partial t} M(t,x)_2\right) - 2\left(\frac{\partial}{\partial x} M(t,x)_1\right)\left(\frac{\partial}{\partial x} M(t,x)_2\right)\right)\beta + \left(\left(\frac{\partial}{\partial t} M(t,x)_1\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_1\right)^2\right)\alpha.$$

Hence :

$$\left(\left(\frac{\partial}{\partial t} M(t,x)_2\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_2\right)^2 + 1\right)\gamma + \left(2\left(\frac{\partial}{\partial t} M(t,x)_1\right)\left(\frac{\partial}{\partial t} M(t,x)_2\right) - 2\left(\frac{\partial}{\partial x} M(t,x)_1\right)\left(\frac{\partial}{\partial x} M(t,x)_2\right)\right)\beta + \left(\left(\frac{\partial}{\partial t} M(t,x)_1\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_1\right)^2 - 1\right)\alpha = 0 \quad \forall\,(\alpha,\beta,\gamma)$$

Conclusion :

$$\left(\frac{\partial}{\partial t} M(t,x)_1\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_1\right)^2 = 1,$$

$$\left(\frac{\partial}{\partial t} M(t,x)_2\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_2\right)^2 = -1 \quad \textit{et}$$

$$\left(\frac{\partial}{\partial t} M(t,x)_1\right)\left(\frac{\partial}{\partial t} M(t,x)_2\right) - \left(\frac{\partial}{\partial x} M(t,x)_1\right)\left(\frac{\partial}{\partial x} M(t,x)_2\right) = 0 .$$

As $M(t,x)_1 = F1(t+x) + F2(t-x)$ *et* $M(t,x)_2 = G1(t+x) + G2(t-x)$ *:*

$$\left(\frac{\partial}{\partial t} M(t,x)_1\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_1\right)^2 = 4\, F1'(t+x) \cdot F2'(t-x) ,$$

and

$$\left(\frac{\partial}{\partial t} M(t,x)_2\right)^2 - \left(\frac{\partial}{\partial x} M(t,x)_2\right)^2 = 4\, G1'(t+x) \cdot G2'(t-x) .$$

Notice : When M is linear,

$$4\, F1'(t+x) \cdot F2'(t-x) = 4 \cdot \frac{(m_{1,1} + m_{1,2})}{2} \cdot \frac{(m_{1,1} - m_{1,2})}{2} = m_{1,1}^2 - m_{1,2}^2$$

$$4\, G1'(t+x) \cdot G2'(t-x) = 4 \cdot \frac{(m_{2,1} + m_{2,2})}{2} \cdot \frac{(m_{2,1} - m_{2,2})}{2} = m_{1,2}^2 - m_{2,2}^2$$

And

$$\left(\frac{\partial}{\partial t} M(t,x)_1\right)\left(\frac{\partial}{\partial t} M(t,x)_2\right) - \left(\frac{\partial}{\partial x} M(t,x)_1\right)\left(\frac{\partial}{\partial x} M(t,x)_2\right)$$

$$= \left(\frac{\partial}{\partial t} (F1(t+x) + F2(t-x))\right)\left(\frac{\partial}{\partial t} (G1(t+x) + G2(t-x))\right)$$

$$- \left(\frac{\partial}{\partial x} (F1(t+x) + F2(t-x))\right)\left(\frac{\partial}{\partial x} (G1(t+x) + G2(t-x))\right);$$

$$= 2\, (F1'(t+x) G2'(t-x) + F2'(t-x)\, G1'(t+x))$$

Notice : When M is linear,

$$2 \cdot \left(\frac{(m_{1,1} + m_{1,2})}{2} \cdot \frac{(m_{2,1} - m_{2,2})}{2} + \frac{(m_{1,1} - m_{1,2})}{2} \cdot \frac{(m_{2,1} + m_{2,2})}{2}\right)$$

$$= m_{1,1} m_{2,1} - m_{1,2} m_{2,2} .$$

We can conclude :

As $M(t,x)$ *can be written as* $M(t,x) = \begin{bmatrix} M(t,x)_1 \\ M(t,x)_2 \end{bmatrix} = \begin{bmatrix} F1(t+x) + F2(t-x) \\ G1(t+x) + G2(t-x) \end{bmatrix}$

where ***F1, F2, G1*** *et* ***G2*** *are arbitrary* C^2 *− functions with the conditions :*

$4\, F1'(t+x) \cdot F2'(t-x) = 1, \qquad (1)$

$4\, G1'(t+x) \cdot G2'(t-x) = -1 \qquad (2)$

$F1'(t+x) G2'(t-x) + F2'(t-x)\, G1'(t+x) = 0 \;. (3)$

If in (1) *and* (2) *we put* $t = x$ *and* $t = -x$ *we can see* ***F1', F2', G1'*** *et* ***G2'*** *are constant ,*
then ***F1, F2, G1*** *et* ***G2*** *are affinities .*

If $M(0,0) = (0,0)$ ***M*** *is linear and a matrix of Lorentz .*

Notice :

Why $\mathbf{M}\mathbf{G}{}^{t}\mathbf{M}=\mathbf{G}$ *is the key point ?*

That comes from chain rule :

$$If \quad X' = MX = \begin{bmatrix} m_{1,1}x^1 + m_{1,2}x^2 + m_{1,3}x^3 + m_{1,4}x^4 \\ m_{2,1}x^1 + m_{2,2}x^2 + m_{2,3}x^3 + m_{2,4}x^4 \\ m_{3,1}x^1 + m_{3,2}x^2 + m_{3,3}x^3 + m_{3,4}x^4 \\ m_{4,1}x^1 + m_{4,2}x^2 + m_{4,3}x^3 + m_{4,4}x^4 \end{bmatrix}.$$

we have $\dfrac{\partial x'^a}{\partial x^b} = m_{a,b}$ *and* (*cf* : [1]*p*.95 *et*[4]) :

$$\frac{\partial f(X')}{\partial x^b} = \sum_{a=1}^{4} \frac{\partial f(X')}{\partial x'^a}\,\frac{\partial x'^a}{\partial x^b} = \sum_{a=1}^{4} \frac{\partial f(X')}{\partial x'^a}\, m_{a,b}$$

$$then \quad \frac{\partial f(X')}{\partial x^b} = \frac{\partial f(X')}{\partial x'^1} m_{1,b} + \frac{\partial f(X')}{\partial x'^2} m_{2,b} + \frac{\partial f(X')}{\partial x'^3} m_{3,b} + \frac{\partial f(X')}{\partial x'^4} m_{4,b},$$

$$and \quad \left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right] f(X') = \left[\frac{\partial}{\partial x'^1}, \frac{\partial}{\partial x'^2}, \frac{\partial}{\partial x'^3}, \frac{\partial}{\partial x'^4}\right] \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} f(X'),$$

that implies :

$$\left[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x^1} \\ \dfrac{\partial}{\partial x^2} \\ \dfrac{\partial}{\partial x^3} \\ \dfrac{\partial}{\partial x^4} \end{bmatrix} f(X')$$

$$= \left[\frac{\partial}{\partial x'^1}, \frac{\partial}{\partial x'^2}, \frac{\partial}{\partial x'^3}, \frac{\partial}{\partial x'^4}\right] \begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} {}^{t}\!\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x'^1} \\ \dfrac{\partial}{\partial x'^2} \\ \dfrac{\partial}{\partial x'^3} \\ \dfrac{\partial}{\partial x'^4} \end{bmatrix} f(X')$$

if $\mathbf{M}$ *is a matrix of Lorentz :*

$$\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} {}^{t}\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} \\ m_{4,1} & m_{4,2} & m_{4,3} & m_{4,4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and then :}$$

$$\frac{\partial^2 f(X')}{\partial x^{12}} - \frac{\partial^2 f(X')}{\partial x^{22}} - \frac{\partial^2 f(X')}{\partial x^{32}} - \frac{\partial^2 f(X')}{\partial x^{42}} = \frac{\partial^2 f(X')}{\partial x'^{12}} - \frac{\partial^2 f(X')}{\partial x'^{22}} - \frac{\partial^2 f(X')}{\partial x'^{32}} - \frac{\partial^2 f(X')}{\partial x'^{42}} .$$

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